# ON THE SMOOTH LOCUS OF ALIGNED HILBERT SCHEMES THE $k$-SECANT LEMMA AND THE GENERAL PROJECTION THEOREM

LAURENT GRUSON AND CHRISTIAN PESKINE

Abstract. Let $X$ be a smooth, connected, dimension $n$, quasi-projective variety embedded in $\mathbb{P}^N$. Consider integers $\{k_1, ..., k_r\}$, with $k_i > 0$, and the Hilbert Scheme $H_{\{k_1,...,k_r\}}(X)$ of aligned, finite, degree $\sum k_i$, subschemes of $X$, with multiplicities $k_i$ at points $x_i$ (possibly coinciding). The expected dimension of $H_{\{k_1,...,k_r\}}(X)$ is $2N - 2 + r - (\sum k_i)(N - n)$. We study the locus of points where $H_{\{k_1,...,k_r\}}(X)$ is not smooth of expected dimension and we prove that the lines carrying this locus do not fill up $\mathbb{P}_N$.

## 1. Introduction

Let $C \subset \mathbb{P}^3(\mathbb{C})$ be a smooth curve in the projective complex space . A general projection $p : C \to C_1 \subset \mathbb{P}^2(\mathbb{C})$ has only ordinary double points as singularities. This statement, known as the 3-secant lemma, is composed of three assertions:

1) the tangents to $C$ do not fill up the space,
2) the tacnode, or stationary, or ramified 2-secant lines to $C$ do not fill up the space,
3) the 3-secants to $C$ do not fill up the space.

The proof is classical and easy to explain . We note that 1) is obvious (counting dimensions). If 2) were not true, two tangents would always intersect. Consequently, if $C$ is not a plane curve, all tangents would pass through a point and C would be everywhere ramified over its projection from this point. As for 3), it reduces to 2). Indeed, if every 2-secant to $C$ is a 3-secant to $C$, it is not difficult to check that two tangents always intersect.

It is well known that the double locus $C_2$ of the projection $C_1$ has a natural structure of smooth variety whose ideal in $C_1$ is the conductor. The tangent space to $C_2$ is implicitly described in the 3-secant lemma. Consider $z \in C_2$ and the points $x_1, x_2 \in p^{-1}(z)$, the tangent space to $C_2$ at $z$ is the intersection of the projections of the tangent spaces (lines) to $C$ at $x_1$ and $x_2$ (they intersect transversally).

Before discussing possible generalizations of this result to higher dimensions, let us agree that in this paper a line $L \subset \mathbb{P}^N(\mathbb{C})$ is a $k$-secant to a smooth quasi-projective variety $Z \subset \mathbb{P}^N(\mathbb{C})$ if the scheme $L \cap Z$ is finite of degree $\geq k$.

The 3-secant lemma was first generalized by Z. Ran ([8]) as follows: the $n+2$-secants to a smooth, dimension $n$, projective variety $X \subset \mathbb{P}^N(\mathbb{C})$ fill up a variety of dimension at most $n + 1$.

Recently R. Beheshti and D. Eisenbud improved significantly Ran's lemma (see [3], Theorem 1.5.): they prove that for $k > [n/s] + 1$ (where $[n/s]$ is the integral part of $n/s$), the $k$-secant lines to a smooth, dimension $n$, projective variety $X \subset \mathbb{P}^N(\mathbb{C})$ fill up a variety of dimension at most $n + s$.

Note that if $c = N - n$ and if $k \leq n/(c - 1) + 1$, the $k$-secants to $X$ may fill up the space. In this case, the projection of $X$ from a general point may have points of order $\geq k$ and we need information such as dimension, smoothness, description of tangent spaces concerning the geometric nature of their locus. For example, using the result of Beheshti/Eisenbud, it is clear that the projection (from a general point) of a smooth, dimension 6, variety $X \subset \mathbb{P}^9(\mathbb{C})$ has no points of order 5. We would like to know the dimension and the singular locus of the loci of points of order $k$, for $2 \leq k \leq 4$ for this projection.

Here is our first result.

*Date*: October 10, 2010.

**Theorem 1.1.** *(General Projection Theorem) Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n$ and codimension $c = N - n$, and let $\pi : X \to X_1 \subset \mathbb{P}^{N-1}$ be a projection from a general point of $\mathbb{P}^N$.*

*1) For $k = k_1 + ... + k_r$, with $k_i > 0$, let $X_{\{k_1,...,k_r\}} \subset X_1$ be the subscheme formed by points $x \in X_1$ such that $\pi^{-1}(x)$ contains $r$ points $\{x_1, ..., x_r\}$ (possibly coinciding) with multiplicity $\geq k_i$ in $x_i$. Then*

*1) The scheme $X_{\{k_1,...,k_r\}}$ has pure dimension $N - 1 - \sum_{i=1}^{r}(k_i c - 1) = N - 1 + r - kc$ (the empty set has all dimensions).*

*2) The singular locus of $X_{\{k_1,...,k_r\}}$ is $X_{\{k_1,...,k_r,1\}}$.*

*3) The normalization $\tilde{X}_{\{k_1,...,k_r\}}$ of $X_{\{k_1,...,k_r\}}$ is smooth.*

**Remarks 1.2.** *1) Be careful, if $x_i$ and $x_j$ do coincide, the multiplicity of $\pi^{-1}(x)$ at the point $x_i = x_j$ has to be $\geq (k_i + k_j)$.*

*2) Please note the following special case of this theorem.*

*When $k_i = 1$ for all $i$, the scheme $X_k = X_{\{1,...,1\}} \subset X_1$ formed by points of multiplicity $\geq k$ of $X_1$ has dimension $N - 1 - k(c - 1)$. The singular locus of $X_k$ is $X_{k+1}$ and the normalization $\tilde{X_k}$ of $X_k$ is smooth.*

It is perhaps worthwhile to emphasize here that the main difficulties to generalize the 3-secant lemma to any dimension appear when the tangent spaces to $X$ fill up the ambient space.

Indeed, suppose they don't. Then the fiber of a point $x \in X_k$ is reduced. By a simple computation in the Grassmann variety $G(1, N)$, one sees that $X_k$ is smooth of dimension $N - 1 - k(c - 1)$ at a point $x$ if and only if the fiber of $x$ has degree $k$ and the projections of the tangent spaces to $X$ at the $k$ distinct points of the fiber are in relatively general position, in which case the tangent space to $X_k$ at $x$ is their intersection. Assuming they are not, there is a corresponding special configuration of linear spaces contained in the Segre $\mathbb{P}_1 \times \mathbb{P}_{N-2}$ whose projective cone is the intersection of $G(1, N)$ with its tangent space at the point (line) corresponding to $x$. Imitating the proof of the 3-secant Lemma, one can prove that if these linear spaces (in $k$ distinct $\mathbb{P}^{N-2}$ of the Segre) are not in relative general position, the projections of any $k - 1$ among them are not in relatively general position. The conclusion of the proof in this case goes through an easy analysis of aligned Hilbert schemes (see section 2 below).

If the tangent spaces to $X$ fill up the ambient space, one can consider the open subset $\hat{X}_k \subset X_k$ formed by points $x \in X_k$ whose fiber is reduced of degree $k$. The same argument proves that $X_k$ is smooth of dimension $N - 1 - k(c - 1)$ at the points of $\hat{X}_k$.

There is no intuitive geometric description of the tangent space to $X_k$ in a point corresponding to a tangent line to $X$ (except for $k = 2$). That is why to prove our main results we have to describe algebraically (and with brutal force) the local equations and the local cotangential equations of $X_k$.

To our knowledge, it was not known that $\hat{X}_k$ is a dense open set in $X_k$. This is implicit in our Theorem 1.1, in particular

$$\hat{X}_k = \emptyset \Rightarrow X_k = \emptyset.$$

Our General Projection Theorem is not detailed enough. We should be more precise about the scheme structure of the closed algebraic set $X_{\{k_1,...,k_r\}}$. The best way to do this, and more generally to clarify our point of view, is to state and prove our result in the language of Hilbert Schemes of aligned points. We know that such Hilbert Schemes are well defined and equipped with an obvious map to the Grassmann variety of lines. The normalization $\tilde{X}_{\{k_1,...,k_r\}}$ is the inverse image of $X_{\{k_1,...,k_r\}}$ in the corresponding Hilbert Scheme. This is explained and described in the following theorem, of which the previous one is clearly a consequence.

From now on, $G(1, N)$ is the Grassmann variety of lines in $\mathbb{P}^N$ and we denote by $\mathcal{I} \subset G \times \mathbb{P}^N$ the incidence variety point/line. We recall that $\mathcal{I}$ is a projective line bundle over $G$ on one hand, and a $(\mathbb{P}^{N-1})$-bundle over $\mathbb{P}^N$ on the other hand.

**Theorem 1.3.** *(Aligned Hilbert Scheme Theorem) Let $X$ be a smooth, connected, dimension $n$, quasi-projective variety embedded in $\mathbb{P}^N$, with $N = n + c$.*

*For $k = k_1 + ... + k_r$, with $k_i > 0$, let $H_{\{k_1,...,k_r\}}(X)$ be the Hilbert scheme of aligned, finite, degree $k$ subschemes of $X$, with multiplicities $k_i$ in points $x_i$ (possibly coinciding). Consider the natural projective line*

*bundle $H_{\{k_1,\ldots,k_r\}}(X)\times_G \mathcal{I}$ over $H_{\{k_1,\ldots,k_r\}}(X)$ and the projection*

$$\theta_{\{k_1,\ldots,k_r\}} : H_{\{k_1,\ldots,k_r\}}(X)\times_G \mathcal{I}\to\mathcal{I}\to\mathbb{P}^N.$$

*Then the general fiber of $\theta_{\{k_1,\ldots,k_r\}}$ is smooth of pure dimension $N-1+r-kc$.*

As in the case of the general projection theorem, the following remarks are important.

**Remarks 1.4.** *1) When $x_i$ and $x_j$ coincide, the multiplicity of a point $h\in H_{\{k_1,\ldots,k_r\}}(X)$ at $x_i=x_j$ has to be $(k_i+k_j)$.*

*2) Please note the following special case of this theorem.*

*When $k_i=1$ for all $i$, if we denote $H_k(X)=H_{\{1,\ldots,1\}}(X)$ the Hilbert scheme of aligned, finite, degree $k$ subschemes of $X$ and $\theta_k : H_k(X)\times_G \mathcal{I}\to\mathcal{I}\to\mathbb{P}_N$, then the general fiber of $\theta_k$ is smooth of dimension $N-1-k(c-1)$.*

As a special case of Theorem 1.3, for $r=1$ and any $k$, we recover a well known result of Mather (see [7]): "higher polar varieties" of a general point with respect to a smooth variety $X$ cut in $X$ a smooth variety of expected dimension.

**Corollary 1.5.** *(Mather) Let $HB_k(X)=H_{\{k\}}(X)\subset\mathcal{I}$ be the Hilbert-Boardmann locus of all $(L,x)\in\mathcal{I}$ such that $L\cap X$ has multiplicity at least $k$ at the point $x$. Consider the natural projective line bundle $HB_k(X)\times_G\mathcal{I}$ and the projection*

$$\theta_{\{k\}} : HB_k(X)\times_G\mathcal{I}\to\mathcal{I}\to\mathbb{P}^{n+c}.$$

*Then the general fiber of $\theta_{\{k\}}$ is smooth of pure dimension $n-(k-1)c=N-kc$.*

We can as well note here that the result of R. Beheshti and D. Eisenbud is recovered as a direct consequence of Theorem 1.3. Indeed, assume $k>[n/s]+1$ (where $[n/s]$ is the integral part of $n/s$). We need to show that the $k$-secant lines to $X$ fill up a variety of dimension at most $n+s$.

For $s\geq c$, there is nothing to prove. If $s=c-1$, this is a special case of our theorem. Assume $s\leq c-2$ and let $L$ be a $k$-secant line of $X$. Consider a projection $X\to\mathbb{P}_{n+s+1}$ whose double locus avoids the finite scheme $L\cap X$. By Theorem 1.3, the $k$-secants to the smooth locus of the image of this projection fill at most a hypersurface in $\mathbb{P}_{n+s+1}$, hence the $k$-secant lines of $X$ near $L$ fill a variety of dimension at most $n+s$ in $\mathbb{P}^N$.

□

Our Aligned Hilbert Scheme Theorem is an easy consequence of the Aligned Ordered Hilbert Scheme Theorem.

The ordered Hilbert schemes $OH_{(k_1,\ldots,k_r)}(X)$ parametrizes finite aligned subschemes $Z\subset X$ supported in an ordered set of points $(x_1,\ldots,x_r)\in X^r$ (not necessarily distinct) and with ordered multiplicities $k_i$ at $x_i$ (note once again that if a point is redundant, for example if $x=x_{i_1}=\ldots=x_{i_s}$, then $Z\subset L\cap X$ must have multiplicity $k_{i_1}+\ldots+k_{i_s}$ at $x$).

Since $OH_{(k_1,\ldots,k_r)}(X)$ is finite and flat over $H_{\{k_1,\ldots,k_r\}}(X)$, it is clear that if $OH_{(k_1,\ldots,k_r)}(X)$ is smooth, then so is $H_{\{k_1,\ldots,k_r\}}(X)$ (the converse is not true). Theorem 1.3 is then a straightforward corollary of the following stronger result.

**Theorem 1.6.** *(Aligned Ordered Hilbert Scheme Theorem) Let $X$ be a smooth, connected, dimension $n$ quasi-projective variety embedded in $\mathbb{P}^N$, with $N=n+c$.*

*For $k=k_1+\ldots+k_r$, with $k_i>0$, let $OH_{(k_1,\ldots,k_r)}(X)$ be the ordered Hilbert scheme of aligned, finite, degree $k$ subschemes of $X$, with (ordered) multiplicities $k_i$ at the ordered points $x_i$ (possibly coinciding). Consider the natural projective line bundle $H_{(k_1,\ldots,k_r)}(X)\times_G\mathcal{I}$ over $H_{(k_1,\ldots,k_r)}(X)$ and the projection*

$$\theta_{(k_1,\ldots,k_r)} : OH_{(k_1,\ldots,k_r)}(X)\times_G\mathcal{I}\to\mathcal{I}\to\mathbb{P}N.$$

*The general fiber of $\theta_{(k_1,\ldots,k_r)}$ is smooth of dimension $N-1+r-kc$.*

The three following sections are devoted to the proof of this theorem.

In section 2 we consider a closed subvariety $\Gamma$ of an affine line $\mathbb{A}^1_{\operatorname{Spec} R}=\operatorname{Spec}(R[z])$ over an affine smooth variety, i.e. $R$ is a finitely generated regular $\mathbb{C}$-algebra. To the morphism

$$\phi : \mathit{\Gamma} = \operatorname{Spec}(R[z]/J) \to \operatorname{Spec} R$$

are associated ordered aligned Hilbert Schemes, that we denote $OH_{(k_1,\dots,k_r)}(\mathit{\Gamma})$ or $OH_{(k_1,\dots,k_r)}(\phi)$, parametrizing subschemes of the fibers of $\phi$ with ordered multiplicities $k_i$ in ordered sets of points of the fibers. Such Hilbert Schemes are equipped with obvious set maps

$$OH_{(k_1,\dots,k_r)}(\phi) = OH_{(k_1,\dots,k_r)}(\mathit{\Gamma}) \to \operatorname{Spec}(R[z_1, ..., z_r]).$$

In Proposition 2.1 we recall that these maps are embeddings which we describe. We state and prove two general technical lemmas that we use repeatedly in the sequel. In particular, Lemma 2.4 describes the local equations and the local cotangential equations of the embedding $OH_{(k_1,\dots,k_r)}(\phi) \subset \operatorname{Spec}(R[z_1, ..., z_r])$. This is elementary calculus.

In section 3, we focus on the case where the base affine variety is an open affine subvariety is a Grassmann variety. More precisely we interpret the last lemma of section 2 in two special cases. On the one hand, when $\operatorname{Spec} R$ is an open set of $G(1,N)$ (the Grassmann variety of lines in $\mathbb{P}^N$), and on the other hand when $\operatorname{Spec} R$ is an open set of $\mathbb{P}^{N-1}(\beta)$ (the Grassmann variety of lines through a point $\beta \in \mathbb{P}^N$).

The proof of Theorem 1.6 is presented in section 4. We follow an induction principle inspired by the classical proof of the 3-secant lemma. The main difficulty stems from the fact that there is no natural geometric description of the tangent space to the Hilbert-Boardmann Scheme $HB_k(X)$ in a general point. We overcome this difficulty by exploiting the "algebraic Segre nature" of a tangent space to the Grassmann variety of lines. This Segre structure is described with all necessary precautions in section 3.

The last section is dedicated to examples, questions and conjectures.

As a conclusion to this introduction, we wish to thank the referees for their constructive remarks and critics.

## 2. The local Aligned Ordered Hilbert Scheme.

In this section $R$ is a regular finitely generated $\mathbb{C}$-algebra. We consider an affine line $\operatorname{Spec} R[z]$ over the affine smooth variety $\operatorname{Spec} R$, a closed subscheme $\mathit{\Gamma} = \operatorname{Spec}(R[z]/J)$ of this affine line and the morphism

$$\phi : \mathit{\Gamma} = \operatorname{Spec}(R[z]/J) \to \operatorname{Spec} R.$$

The aligned ordered Hibert Scheme $OH_{(k_1,\dots,k_r)}(\phi)$ parametrizes subschemes

$$Z \subset \operatorname{Spec} \mathbb{C}[z] = \phi^{-1}(x), \quad x \in \operatorname{Spec} R$$

with support in an ordered set of points $a_1, ..., a_r \in \operatorname{Spec} \mathbb{C}[z]$, and with length (multiplicity) $k_i$ at $a_i$. Two points $a_i$ and $a_j$ may coincide, as long as the finite scheme $Z$ has length multiplicity $k_i + k_j$ in $a_i = a_j$.

When $\mathit{\Gamma}$ is a hypersurface, i.e. when $\mathit{\Gamma} = \operatorname{Spec}(R[z]/(g))$, where $g = g(z)$ is a polynomial, we often write $OH_{(k_1,\dots,k_r)}(g)$ instead of $OH_{(k_1,\dots,k_r)}(\phi)$.

The following proposition is well known to anyone familiar with aligned Hilbert schemes (see for example [4] or [5]). For a reader who is not, the best is to admit 1), which by the way explains why we prefer the ordered Hilbert scheme to the nonordered one.

**Proposition 2.1.** *1) We have*

$$OH_{(k_1,\dots,k_r)}(g) = \operatorname{Spec}(R[z_1, ..., z_r]/(h_0, ..., h_{k-1})), \ k = \sum k_i,$$

*where the polynomials $h_l \in R[z_1, ..., z_r]$ are defined for $l = 0, ..., k-1$ by*

$$g(z) \equiv \sum_0^{k-1} h_l(z_1, ..., z_r) z^l \mod(\prod_1^r (z - z_i)^{k_i}).$$

*2) If $\mathit{\Gamma} = \operatorname{Spec}(R[z]/(g_1, ..., g_c))$, then*

$$OH_{(k_1,\dots,k_r)}(\phi) = \cap_1^c H_{k_1,\dots,k_r}(g_t).$$

*3) If $R'$ is a finitely generated regular $R$-algebra and if $\phi' = \phi \otimes_R R' : \mathrm{Spec}\ A \otimes_R R' \to \mathrm{Spec}\ R'$, then*

$$OH_{(k_1,\dots,k_r)}(\phi') = OH_{(k_1,\dots,k_r)}(\phi) \times_{\mathrm{Spec}\, R} (\mathrm{Spec}\, R').$$

We observe (with pleasure) that $OH_1(g) \simeq \mathrm{Spec}(R[z]/(g(z))$ and more generally $OH_1(\phi) \simeq \Gamma$.

From now on, we shall pay a particular attention to the case when $\Gamma = \mathrm{Spec}\ (R[z]/J)$ is a smooth complete intersection in $\mathrm{Spec}\ R[z]$.

We begin with obvious remarks.

**Remarks 2.2.** *1) The expected dimension of $OH_{(k_1,\dots,k_r)}(g)$ is $\dim R + r - k = \dim\ R + \sum_1^r (1 - k_i)$.*

*2) When $R[z]/J$ is a complete intersection of codimension $c$ in $R[z]$, the expected dimension of $OH_{(k_1,\dots,k_r)}(\phi)$ is $\dim\ R + r - ck$.*

We note (with great pleasure once again) that since $OH_1(\phi) \simeq \Gamma$ it is clear that when $\Gamma$ is smooth, so is the ordered Hilbert Scheme $OH_1(\phi)$. This obvious remark will be the starting point of the proof by induction of Theorem 1.6.

The following result will prove to be an important technical tool in the proof of our main theorem. To be more precise, it will allow us, when necessary, to work with points $(x, a_1, \dots, a_r) \in OH_{(k_1,\dots,k_r)}(\phi) \subset \mathrm{Spec}\ R[z_1, \dots, z_r]$, with $x \in \mathrm{Spec}\, R$ and $a_i \in \mathbb{C}$, such that $a_i \neq a_j$ for $i \neq j$.

**Lemma 2.3.** *Assume $R[z]/J$ is a complete intersection of codimension $c$ in $R[z]$.*

*Consider a point $(x, a_1, \dots, a_r) \in OH_{(k_1,\dots,k_r)}(\phi)$ (with $x \in \mathrm{Spec}\, R$ and $a_1, \dots, a_r \in \mathrm{Spec}\ \mathbb{C}[z] = \phi^{-1}(x)$). Assume that there exist $1 \leq s < t \leq r$ such that $a_s = a_t$, in other words that*

$$(x, a_1, \dots, a_s, \dots, a_{t-1}, a_{t+1}, \dots, a_r) \in OH_{(k'_1,\dots,k'_{r-1})}(\phi),$$

*with $k_i = k'_i$ for $i < s$ and $s < i < t$, $k'_s = k_s + k_t$ and $k'_{i-1} = k_i$ for $i > t$.*

*Then $OH_{(k_1,\dots,k_r)}(\phi)$ is smooth of expected dimension at $(x, a_1, \dots, a_r)$ if and only if $OH_{(k'_1,\dots,k'_{r-1})}(\phi)$ is smooth of expected dimension at $(x, a_1, \dots, a_s, \dots, a_{t-1}, a_{t+1}, \dots, a_r)$.*

*Proof.* For the sake of simplicity, we assume

$$R[z]/J = R[z]/(g), \quad s = 1, \quad r = k = t = 2, \quad k_1 = k_2 = 1, \quad a_1 = a_2 = 0.$$

Put $g(z) = \sum_{i \geq 0} \alpha_i z^{d-i}$. Let $\mathcal{M}$ be the maximal ideal of $R$ corresponding to the point $x \in \mathrm{Spec}\ R$. Since 0 is a point of multiplicity $\geq 2$ in the fiber of $x$, we have $\alpha_d, \alpha_{d-1} \in \mathcal{M}$.

$$g(z) \equiv (\alpha_{d-1} + \alpha_{d-2}(z_1 + z_2))z + \alpha_d \mod((z - z_1)(z - z_2) + (\mathcal{M}, z_1, z_2)^2)R[z_1, z_2][z],$$

and

$$g(z) \equiv (\alpha_{d-1} + 2\alpha_{d-2}z_1)z + \alpha_d \mod((z - z_1)^2 + (\mathcal{M}, z_1)^2)R[z_1][z].$$

Two cases occur (depending of the multiplicity of the root 0 of the image of $g$ in $(R/\mathcal{M})[z] = \mathbb{C}[z]$).

1) If the multipicity is precisely 2, i.e. if $\alpha_{d-2} \notin \mathcal{M}$, then

$$\begin{aligned} OH_{1,1}(g) &\text{ is smooth of expected dimension } \dim R \Leftrightarrow \alpha_d \notin \mathcal{M}^2 \\ &\Leftrightarrow OH_2(g) \text{ is smooth of expected dimension } \dim R - 1. \end{aligned}$$

2) If the multipicity is $> 2$, i.e if $\alpha_{d-2} \in \mathcal{M}$, then

$$\begin{gathered} OH_{1,1}(g) \text{ is smooth of expected dimension } \dim R \\ \Leftrightarrow \alpha_{d-1} \text{ and } \alpha_d \text{ are transverse in } \mathcal{M}/\mathcal{M}^2 \\ \Leftrightarrow OH_2(g) \text{ is smooth of expected dimension } \dim R - 1. \end{gathered}$$

□

Our next result describes explicitly the local equations and the local cotangential equations of the ordered Hilbert Scheme at a point

$$(x, a_1, ..., a_r) \in OH_{(k_1,...,k_r)}(g) \subset \text{Spec } R[z_1, ..., z_r],$$

with $x \in \operatorname{Spec} R$ and $a_i \in \mathbb{C}$. It will be used more than once (and without thinking twice).

From now on, we denote by $g^{(s)}(z)$ the derivative of order $s$ of the function $g(z)$ for $s \geq 0$. The convention $g^{(-1)}(z) = 0$ will prove to be useful later on.

**Lemma 2.4.** *Assume that $\Gamma = \operatorname{Spec}(R[z]/(g))$ is a hypersurface and consider a point $(x, a_1, ..., a_r) \in OH_{(k_1,...,k_r)}(g)$ supported in the fiber $\phi^{-1}(x)$, with $x \in \operatorname{Spec} R$ and $a_i \in \mathbb{C}$, such that $a_i \neq a_j$ for $i \neq j$.*

*1) The local equations of $OH_{(k_1,...,k_r)}(g) \subset \text{Spec } R[z_1, ..., z_r]$, at $(x, a_1, ..., a_r)$, are*

$$g^{(s)}(z_i)/s!, \quad i = 1, ..., r, \quad 0 \leq s < k_i.$$

*2) If $\mathcal{M}$ is the maximal ideal of $R$ corresponding to the point $x \in$ Spec $R$, the local cotangential equations of $OH_{(k_1,...,k_r)}(g) \subset \operatorname{Spec} R[z_1, ..., z_r]$ at $(x, a_1, ..., a_r)$ in the cotangent space*

$$(\mathcal{M}, (z_1 - a_1), ..., (z_r - a_r))/(\mathcal{M}, (z_1 - a_1), ..., (z_r - a_r))^2$$

*of $\operatorname{Spec} R[z_1, ..., z_r]$ in the point $(x, a_1, ..., a_r)$ , are the classes of the $r(\sum k_i)$ elements*

$$g^{(s)}(a_i) \quad 0 \leq s < k_i - 1, \qquad g^{(k_i-1)}(a_i) + (z_i - a_i)g^{(k_i)}(a_i)$$

*for $i = 1, ..., r$.*

*Proof.* 1) is an obvious consequence of the Taylor expansions $g(z) = \sum_{s>0}(g^{(s)}(z_i)/s!)(z - z_i)^s$.

2) is easily deduced from the relation $g^{(s)}(z_i) \equiv g^{(s)}(a_i) + (z_i - a_i)g^{(s+1)}(a_i) \mod (z_i - a_i)^2$. □

The following remarks (using the same notations as in the lemma) are important.

**Remarks 2.5.** *1) For $s < k_i - 1$, the classes*

$$cl(g^{(s)}(a_i)) \in (\mathcal{M}, (z_1 - a_1), ..., (z_r - a_r))/(\mathcal{M}, (z_1 - a_1), ..., (z_r - a_r))^2$$

*are in the vector subspace $\mathcal{M}/\mathcal{M}^2$*

*2) If $g^{k_i}(a_i) \in \mathcal{M}$, i.e. if $cl(g(z)) \in (R/\mathcal{M})[z]$ has multiplicity $> k_i$ at $a_i$, then*

$$cl(g^{(k_i-1)}(a_i) + (z_i - a_i)g^{(k_i)}(a_i)) = cl(g^{(k_i-1)}(a_i)) \in \mathcal{M}/\mathcal{M}^2.$$

*3) If $g^{(k_i)}(a_i) \notin \mathcal{M}$, i.e. the order of $g(z)$ at the point $a_i$ of the special fiber is precisely $k_i$, then*

$$cl(g^{(k_i-1)}(a_i) + (z_i - a_i)g^{(k_i)}(a_i)) \notin \mathcal{M}/\mathcal{M}^2.$$

## 3. The local Aligned Ordered Hilbert Scheme over a Grassmann Variety.

In the first part of this section, we assume that Spec $R \simeq \mathbb{A}^{2N-2}$ is an affine open set of the Grassmann variety $G(1, N)$.

We consider an affine line $L \subset \mathbb{A}^N = \operatorname{Spec} \mathbb{C}[x_1, ..., x_{N-1}, z]$, with equations $x_1 = ... = x_{N-1} = 0$.

Let $\operatorname{Spec} R = \text{Spec } \mathbb{C}[u_1, ..., u_{N-1}, v_1, ..., v_{N-1}]$ be such that

- the line $L$ corresponds to the origin $(0, ..., 0) \in$ Spec $R$,
- the local system of parameters $u_i, v_i$ (for $1 \leq i \leq N - 1$) of $G(1, N)$ and the indeterminate $z$ parametrizing the canonical affine line over $\operatorname{Spec} R$ verify the relations

$$x_i = u_i z + v_i, \text{ for } i = 1, ..., N - 1.$$

The canonical inclusions

$\mathbb{C}[x_1,...,x_{N-1},z]=\mathbb{C}[u_1z+v_1,...,u_{N-1}z+v_{N-1},z]\subset\mathbb{C}[u_1,...,u_{N-1},v_1,...,v_{N-1},z]\supset\mathbb{C}[u_1,...,u_{N-1},v_1,...,v_{N-1}]$.
induce the morphisms $\pi$ and $\psi$ in the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[d, equal] & \operatorname{Spec} R[z] \arrow[l] \arrow[r] \arrow[d, equal] & \mathbb{A}^N \arrow[d, equal] \\
\operatorname{Spec}\mathbb{C}[u_1,...,u_{n-1},v_1,...,v_{n-1}] \arrow[d, hook] & \operatorname{Spec}\mathbb{C}[u_1,...,u_{n-1},v_1,...,v_{n-1},z] \arrow[l, "\psi"'] \arrow[r, "\pi"] \arrow[d, hook] & \operatorname{Spec}\mathbb{C}[x_1,...,x_{N-1},z] \arrow[d, hook] \\
G(1,N) & \mathcal{I} \arrow[l] \arrow[r] & \mathbb{P}^N.
\end{tikzcd}$$

In the affine space $\mathbb{A}^N=\operatorname{Spec}\mathbb{C}[x_1,...,x_{N-1},z]$, we consider now an irreducible complete intersection $Y$ and a finite scheme $Z\subset Y\cap L$ supported in $r$ distinct ordered points $(0,...,0,a_i)\in Y\cap L\subset\operatorname{Spec}\mathbb{C}[x_1,...,x_{N-1},z]$ and with multiplicity $k_i$ at $(0,...,0,a_i)$. We assume that $Y$ is smooth at all points of $Z$.

Let $c$ be the codimension of $Y$ in $\mathbb{A}^N$. We can find a system of $c$ hypersurfaces $G_i\subset\mathbb{A}^N$ defined by polynomials $g_i\in\mathbb{C}[x_1,...,x_{N-1},z]$ such that

$$\cap G_i=Y,$$

$$Y\cap L=G_1\cap L,$$

$$L\subset G_s \quad 2\le s\le c.$$

We denote by $\Gamma=\pi^{-1}(Y)\subset\operatorname{Spec}\mathbb{C}[u_1,...,u_{n-1},v_1,...,v_{n-1},z]=\operatorname{Spec}R[z]$ the inverse image of $Y$. It is cut out by the equations $g_s(u_iz+v_i,z)=0\in R[z]$, for $s=1,...,c$. To the morphism

$$\phi:\Gamma=\operatorname{Spec}R[z]/(g_1,...,g_c)\to\operatorname{Spec}R$$

is associated the ordered aligned Hilbert Scheme

$$OH_{(k_1,\dots,k_r)}(\phi)=\cap_{s=1}^{c}OH_{(k_1,\dots,k_r)}(g_s)\subset\operatorname{Spec}R[z_1,...,z_r].$$

We intend to study the tangent space of this Hilbert Scheme at the point $\{Z\}\in OH_{(k_1,\dots,k_r)}(\phi)$. We recall that the $r$ distinct points of $(0,...,0,a_i)\in Y\cap L\subset\operatorname{Spec}\mathbb{C}[x_1,...,x_{N-1},z]$ are smooth in $Y$. As a consequence, we note that $\Gamma$ is smooth at the $r$ points $(0,....,0,a_i)\in\operatorname{Spec}\mathbb{C}[u_i,v_j,z]$.

We begin with describing the equations and the cotangential equations of $OH_{(k)}(g)\subset$ Spec $R[z_1]$ in a neighborhood of $(o,a)$ for a polynomial $g(z)\in\mathbb{C}[x_i,z_1]$. We write $z$ for $z_1$. The proof of the following lemma is straightforward (essentially contained in the statement).

**Lemma 3.1.** *Consider $g(z)\in\mathbb{C}[x_i,z]$ with $i=1,...,N-1$ and $g=g(u_iz+v_i,z)\in R[z]$, with $i=1,...,N-1$. We assume that $cl(g)\in(R/\mathcal{M})[z]$ has multiplicity $\ge k$ at the point $a\in\mathbb{C}$.*

*1) There exists a unique decomposition*

$$g(z)\equiv p(z)+\sum(u_iz+v_i)q_i(z)\mod \mathcal{M}^2R[z],\quad p,q_i\in\mathbb{C}[z],\quad p\in(z-a)^k\mathbb{C}[z],$$

*2) It induces a decomposition (we recall the convention $q^{(-1)}(z)=0$)*

$$g^{(s)}(z)\equiv p^{(s)}(z)+\sum u_iq_i^{(s-1)}(z)+\sum_i(u_iz+v_i)q_i^{(s)}(z)\mod\mathcal{M}^2R[z]$$

*and decompositions*

$$g^{(s)}(a) \equiv \sum_i u_i q_i^{(s-1)}(a) + \sum_i (u_i a + v_i) q_i^{(s)}(a) \mod \mathcal{M}^2 R[z], \quad s < k-1,$$

$$g^{(k-1)}(a) + (z-a) g^{(k)}(a) \equiv$$
$$(z-a)p^{(k)}(a) + \sum_i u_i q_i^{(k-2)}(a) + \sum_i (u_i a + v_i) q_i^{(k-1)}(a) \mod (\mathcal{M}, (z-a))^2.$$

Note here (once again) that if $g$ has multiplicity $> k$ at $a$, then $p^{(k)}(a) \in (z-a)$ and

$$(z-a)p^{(k)}(a) \in (\mathcal{M}, (z-a))^2.$$

This is why we introduce the following unpleasant convention (notation).

*If $e_j$ is the multiplicity of $g_1$ at the point $a_j$, we put $h_j = k_j$ if $e_j > k_j$, and $h_j = k_j - 1$ if $e_j = k_j$.*

In order to apply Lemma 3.1 to the polynomials $g_s(z)$, we consider the unique decompositions

$$g_1(z) \equiv p(z) + \sum (u_i z + v_i) q_{1,i}(z) \mod \mathcal{M}^2 R[z], \quad p, q_{1,i} \in \mathbb{C}[z], \quad p \in \cap_j (z-a_j)^{k_j} \mathbb{C}[z],$$

$$g_t(z) \equiv \sum (u_i z + v_i) q_{t,i}(z) \mod \mathcal{M}^2 R[z], \quad q_{t,i} \in \mathbb{C}[z], \quad t > 1.$$

These decompositions will play an important part in the proof of Theorem 1.6. We choose to underline here the following Proposition which is a straightforward consequence of Lemma 2.4 and Lemma 3.1.

**Proposition 3.2.** *The aligned ordered Hilbert Scheme $OH_{(k_1,...,k_r)}(\phi)$ is smooth of expected dimension $2N-2+r-ck$ at the point $\{Z\} = (o, a_1, ..., a_r) \in \operatorname{Spec} R[z_1, ..., z_r]$, where $a_i \neq a_j$ for $i \neq j$, if and only if the following elements of $\mathcal{M}/\mathcal{M}^2$ are linearly independent.*

$$\sum_i u_i q_{1,i}^{(s-1)}(a_j) + \sum_i (u_i a_j + v_i) q_{1,i}^{(s)}(a_j), \quad j = 1, ..., r, \quad 0 \le s \le h_j - 1,$$

$$\sum_i u_i q_{t,i}^{(s-1)}(a_j) + \sum_i (u_i a_j + v_i) q_{t,i}^{(s)}(a_j), \quad t > 1, \quad j = 1, ..., r, \quad 0 \le s \le k_j - 1.$$

In the second part of this section $\operatorname{Spec} R \simeq \mathbb{A}^{N-1}$ is an affine open set of the Grassmann variety $\mathbb{P}^{N-1}(\beta)$ parametrizing the lines of $\mathbb{P}^N$ through a point $\beta \in \mathbb{P}^N$.

More precisely, from here we fix a point $\beta = (0, ..., 0, b)$, general in the line $L \subset \mathbb{A}^N$. We recall that $\operatorname{Spec} \mathbb{C}[u_i, v_j]$ is an affine open set in $G(1, N)$. The intersection of this open set with the closed subvariety $\mathbb{P}^{N-1}(\beta) \subset G(1, N)$ (the lines through $\beta$) is

$$\operatorname{Spec} \mathbb{C}[u_i, v_j]/(u_i b + v_i).$$

We put

$$R_b = R/(u_i b + v_i)$$

and we denote by $\bar{u}_i$ and $\bar{v}_i$ the classes of $u_i$ and $v_i$ in $R_b$. The relations $\bar{v}_i = -b\bar{u}_i$ need no comment and $\bar{u}_i$ is a system of generators (regular parameters) of the maximal ideal

$$\mathcal{M}_b = \mathcal{M}/(u_i b + v_i) \subset R_b.$$

In the inverse image of $\operatorname{Spec} R_b$, in the incidence variety point/line, we consider, as earlier, the affine open set $\operatorname{Spec} R_b[z]$. This is an open affine variety in the blowing-up $\tilde{\mathbb{P}}^N$ of $\mathbb{P}^N$ at the point $\beta$. We observe now the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} R_b \arrow[d, equal] & \operatorname{Spec} R_b[z] \arrow[l] \arrow[r] \arrow[d, equal] & \mathbb{A}^N \arrow[d, equal] \\
\operatorname{Spec} \mathbb{C}[\bar{u_1},...,\bar{u}_{n-1}] \arrow[d, hook] & \operatorname{Spec} \mathbb{C}[\bar{u_1},...,\bar{u}_{n-1},z] \arrow[l, "\psi_\beta"'] \arrow[r, "\pi_\beta"] \arrow[d, hook] & \operatorname{Spec} \mathbb{C}[x_1,...,x_{N-1},z] \arrow[d, hook] \\
\mathbb{P}^{N-1}(\beta) & \tilde{\mathbb{P}}^N \arrow[l] \arrow[r] & \mathbb{P}^N.
\end{tikzcd}$$

We note that $\pi_\beta$ is the blowing-up of the point $(0,...,0,b) \in \operatorname{Spec} \mathbb{C}[x_1,...,x_{N-1},z]$.

We recall that $L \subset \mathbb{A}^N$ is the affine line with equations $x_i = 0$ with $i = 1,...,N-1$. Its inverse image in $\operatorname{Spec} R_b[z]$ is cut out by the equations

$$0 = x_i = \bar{u}_i z + \bar{v}_i = \bar{u}_i(z-b), \quad i = 1,...,N-1.$$

We also recall that $Y \subset \mathbb{A}^N$ is the complete intersection of $c$ hypersurfaces $G_i \subset \mathbb{A}^N$ defined by polynomials $g_i \in \mathbb{C}[x_1,...,x_{N-1},z]$ such that

$$Y \cap L = G_1 \cap L,$$

$$L \subset G_s, \quad 2 \leq s \leq c.$$

The inverse image (proper transform) $\Gamma_b = \pi_\beta^{-1}(Y) \subset \operatorname{Spec} R_b[z]$ is cut out by the $c$ equations

$$g_s(\bar{u}_i(z-b), z) = 0, \quad s = 1,...,c.$$

We put $g_{s,b} = g_s(\bar{u}_i(z-b), z) \in R_b[z]$, denote $R_b[z]/(g_{1,b},...,g_{c,b}) = R[z]/(g_1,...,g_c) \otimes_R R_b$ and consider the morphism

$$\phi_b : \Gamma_b = \operatorname{Spec}\ R_b[z]/(g_{1,b},...,g_{c,b}) \subset \operatorname{Spec} R_b[z] \to \operatorname{Spec} R_b.$$

We intend to study the tangent space of the Hilbert Scheme at the point

$$\{Z\} \in OH_{(k_1,...,k_r)}(\phi_b) = \cap_{s=1}^{c} OH_{(k_1,...,k_r)}(g_{s,b}) \subset \operatorname{Spec}\ R_b[z_1,...,z_r].$$

Let us recall that the $r$ distinct points of $(0,...,0,a_i) \in Y \cap L \subset \operatorname{Spec} \mathbb{C}[x_1,...,x_{N-1},z]$ are smooth in $Y$. As a consequence, we note that $\Gamma_b$ is smooth at the $r$ points $(0,....,0,a_i) \in \operatorname{Spec} \mathbb{C}[\bar{u}_i,z]$.

The cartesian diagram

$$\begin{tikzcd}
\Gamma_b \arrow[r, "\phi_b"] \arrow[d] & \operatorname{Spec} R_b \arrow[d] \\
\Gamma \arrow[r, "\phi"] & \operatorname{Spec} R
\end{tikzcd}$$

and Proposition 2.1 3) imply the following

**Proposition 3.3.**

$$OH_{(k_1,...,k_r)}(\phi_b) = OH_{(k_1,...,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec} R_b).$$

Next we intend to give explicit necessary and sufficient conditions for the smoothness of $OH_{(k_1,...,k_r)}(\phi_b)$ at $(o,a_1,...,a_r)$. As in the preceding case, we begin by describing the local equations (in a neighborhood of a point $(0,a)$) of the Hilbert Scheme $OH_{(k)}(g_b) \subset \operatorname{Spec}\ R_b[z]$, where $g_b$ is of the forme $g_b(z) = g(\bar{u}_i(z-b), z)$, with $g \in (\mathbb{C}[x_1,...,x_{N-1},z]$. Lemma 3.1 specializes immediately in the following way:

**Lemma 3.4.** *Consider a polynomial $g(x_i, z) \in \mathbb{C}[x_i, z]$, with $i = 1, ..., N-1$, and $g_b = g(\bar{u}_i(z-b), z) \in R_b[z]$, with $i = 1, ..., N-1$. Assume that the polynomial $cl(g_b) \in (R_b/\mathcal{M}_b)[z]$ has multiplicity $\geq k$ at the point $a \in \mathbb{C}$.*

*The unique decomposition (as before, we follow the convention $q_i^{(-1)} = 0$)*

$$g_b(z) \equiv p(z) + \sum \bar{u}_i(z-b)q_i(z) \mod \mathcal{M}^2 R[z], \quad p, q_i \in \mathbb{C}[z], \quad p \in (z-a)^k\mathbb{C}[z],$$

*induces, for all $s$, a decomposition*

$$g_b^{(s)}(z) \equiv p^{(s)}(z) + \sum_i \bar{u}_i q_i^{(s-1)}(z) + \sum_i \bar{u}_i(z-b)q_i^{(s)}(z) \mod \mathcal{M}_b^2 R_b[z].$$

*and decompositions*

$$g_b^{(s)}(a) \equiv \sum_i \bar{u}_i q_i^{(s-1)}(a) + \sum_i \bar{u}_i(a-b)q_i^{(s)}(a) =$$

$$\sum_i \bar{u}_i[q_i^{(s-1)}(a) + (a-b)q_i^{(s)}(a)] \mod \mathcal{M}_b^2 R_b[z], \quad s < k-1,$$

$$g_b^{(k-1)}(a) + (z-a)g_b^{(k)}(a) \equiv$$

$$(z-a)p^{(k)}(a) + \sum_i \bar{u}_i q_i^{(k-2)}(a) + \sum_i \bar{u}_i(a-b)q_i^{(k-1)}(a) \mod (\mathcal{M}_b, (z-a))^2.$$

We recall that that $g_{t,b}(z) \in \mathcal{M}_b R_b[z]$ for $t > 1$. As an immediate consequence, we get the following result (to be compared with Proposition 3.2):

**Proposition 3.5.** *The aligned ordered Hilbert Scheme $OH_{(k_1,...,k_r)}(\phi_b)$ is smooth of expected dimension $N - 1 + r - kc$ at the point $\{Z\} = (o, a_1, ..., a_r) \in \operatorname{Spec} R[z_1, ..., z_r]$ with $a_i \neq a_j$ for $i \neq j$ if and only if the following elements of $\mathcal{M}_b/\mathcal{M}_b^2$ are linearly independent:*

$$\sum_i \bar{u}_i[q_{1,i}^{(s-1)}(a_j) + (a_j - b)q_{1,i}^{(s)}(a_j)], \quad j = 1, ..., r, \quad 0 \leq s \leq h_j - 1,$$

$$\sum_i \bar{u}_i[q_{t,i}^{(s-1)}(a_j) + (a_j - b)q_{t,i}^{(s)}(a_j)], \quad t > 1, \quad j = 1, ..., r, \quad 0 \leq s \leq k_j - 1.$$

□

Finally in this section, we observe that, for $t$ and $j$ fixed and $b \neq a_j$, the vector subspaces of $\mathcal{M}_b/\mathcal{M}_b^2$, generated by

$$\sum_i \bar{u}_i[q_{1,i}^{(s-1)}(a_j) + (a_j - b)q_{1,i}^{(s)}(a_j)]), \quad 0 \leq s \leq h_j - 1, \quad h_j - 1,$$

and

$$\sum_i \bar{u}_i[q_{t,i}^{(s-1)}(a_j) + (a_j - b)q_{t,i}^{(s)}(a_j)]), \quad 0 \leq s \leq k_j - 1, \quad t > 1,$$

on the one hand, and

$$\sum_i \bar{u}_i q_{1,i}^{(s)}(a_j), \quad 0 \leq s \leq h_j - 1,$$

and

$$\sum_i \bar{u}_i q_{t,i}^{(s)}(a_j), \quad 0 \leq s \leq k_j - 1, \quad t > 1,$$

on the other hand, coincide with each other. This proves the equivalence 1) $\Leftrightarrow$ 2) in the following Corollary (of the previous Proposition):

**Corollary 3.6.** *If $b \neq a_j$ for all $j$, the following equivalent conditions are equivalent.*

*1) The aligned ordered Hilbert Scheme $OH_{(k_1,...,k_r)}(\phi_b)$ is smooth of expected dimension $N-1+r-ck$ at the point $\{Z\} = (o, a_1, ..., a_r)$.*

*2) The following elements of $\mathcal{M}_b/\mathcal{M}_b^2$ are linearly independent:*

$$\sum_i \bar{u}_i q_{1,i}^{(s)}(a_j), \quad j = 1, ..., r, \quad 0 \le s \le h_j - 1,$$

$$\sum_i \bar{u}_i q_{t,i}^{(s)}(a_j), \quad t > 1, \quad j = 1, ..., r, \quad 0 \le s \le k_j - 1.$$

*3) The matrix $(q_{t,i}^{(s)}(a_j))$ with $N-1$ rows and $(c-1)k + \sum_j h_j$ columns has maximal rank.*

*Proof.* We have seen 1) $\Leftrightarrow$ 2). The equivalence 2) $\Leftrightarrow$ 3) is an obvious consequence of the fact that $(\bar{u}_i)_i$ is a regular system of generators of $\mathcal{M}_b$. □

Note to conclude this section that condition 3) does not depend on $b$ (this will be a crucial point in the proof (by induction) of Theorem 1.6). In other words, if $\{Z\} = (o, a_1, ..., a_r)$ is a smooth point of $OH_{(k_1,...,k_r)}(\phi_b)$ and if $\beta' = (0, ..., 0, b') \in L$, with $b' \neq a_i, \quad i = 1, ..., r$, then $\{Z\} = (o, a_1, ..., a_r)$ is also a smooth point of $OH_{(k_1,...,k_r)}(\phi_{b'})$.

## 4. Proof of Theorem 1.6

In the previous section, we have studied the configuration of a line $L$, a quasi-projective complete intersection $Y \subset \mathbb{P}^N$ and a general point $\beta \in L$. We studied a finite scheme $Z \subset L \cap Y$, with support in the smooth locus of $Y$ and with multiplicities $(k_1, ..., k_r)$ in $r$ distinct points of $L \cap Y$. This is a point of $OH_{(k_1,...,k_r)}(Y)$. We recall that the inverse images $\Gamma$ and $\Gamma_b$ of $Y$, in the incidence varieties $\mathcal{I}$ and $\mathbb{P}^N(\beta)$, fit in a cartesian diagram

$$\begin{array}{ccc} \Gamma_b & \xrightarrow{\phi_b} & \text{Spec } R_b \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\phi} & \text{Spec } R \end{array}$$

We keep these notations in mind and we come back to the composed projection morphism described in Theorem 1.6

$$\theta_{(k_1,...,k_r)} : OH_{(k_1,...,k_r)}(Y) \times_G \mathcal{I} \to \mathcal{I} \to \mathbb{P}^N.$$

In order to study this morphism in a neighorhood of the locally closed subscheme

$$\{Z\} \times_{\text{Spec } R} \text{Spec } R[z] = (o, a_1, ..., a_r) \times_{\text{Spec } R} \text{Spec } R[z] \subset \text{Spec } R[z_1, ..., z_r, z],$$

we observe the following commutative diagram (where all up vertical arrows are closed immersions):

$$\begin{tikzcd}
\operatorname{Spec} R[z_1,...,z_r] & \operatorname{Spec} R[z_1,...,z_r][z] \arrow[l] \arrow[r] & \operatorname{Spec} R[z] \arrow[r] & \mathbb{A}^N \\
OH_{(k_1,\dots,k_r)}(\phi) \arrow[u] \arrow[d, hook] & OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec} R[z]) \arrow[l] \arrow[r] \arrow[u] \arrow[d, hook] & \operatorname{Spec} R[z] \arrow[r, "p"] \arrow[u, equal] \arrow[d, hook] & \mathbb{A}^N \arrow[u, equal] \arrow[d, hook] \\
OH_{(k_1,\dots,k_r)}(Y) & OH_{(k_1,\dots,k_r)}(Y) \times_G \mathcal{I} \arrow[l] \arrow[r] \arrow[d, equal] & \mathcal{I} \arrow[r] & \mathbb{P}^N \arrow[d, equal] \\
 & OH_{(k_1,\dots,k_r)}(Y) \times_G \mathcal{I} \arrow[r] & \theta_{(k_1,\dots,k_r)} \arrow[r] & \mathbb{P}^N.
\end{tikzcd}$$

We recall here that the equations of $L$ in $\mathbb{A}^N = \operatorname{Spec} \mathbb{C}[x_1,...,x_{N-1},z]$ are $x_1 = ... = x_{N-1} = 0$ and that $\beta = (0,...,0,b) \in L$.

**Proposition 4.1.** $\theta^{-1}_{(k_1,\dots,k_r)}(\beta) \cap (OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec} R[z])) = OH_{(k_1,\dots,k_r)}(\phi_b)$.

*Proof.* We begin with describing the fiber $p^{-1}(\beta) \subset \operatorname{Spec} R[z]$. The maximal ideal of $\mathbb{C}[x_1,...,x_{N-1},z]$ corresponding to $\beta$ is $(x_1,...,x_{N-1},z-b)$ and

$$R[z]/(x_1,...,x_{N-1},z-b) = R[z]/(u_i z + v_i, z-b) = R[z]/(u_i b + v_i, z-b) = R_b.$$

But we have seen (proposition 3.3) that

$$OH_{(k_1,\dots,k_r)}(\phi_b) = OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec} R_b),$$

so the following commutative diagram proves our Proposition:

$$\begin{tikzcd}
\operatorname{Spec} R[z_1,...,z_r] & \operatorname{Spec} R[z_1,...,z_r][z] \arrow[l] \arrow[r] & \operatorname{Spec} R[z] \arrow[r] & \mathbb{A}^N \\
OH_{(k_1,\dots,k_r)}(\phi) \arrow[u] \arrow[d, equal] & OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec} R[z]) \arrow[l] \arrow[r] \arrow[u] & \operatorname{Spec} R[z] \arrow[r] \arrow[u, equal] & \mathbb{A}_N \arrow[u, equal] \\
OH_{(k_1,\dots,k_r)}(\phi) \arrow[d, equal] & OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} (\operatorname{Spec}(R[z] \otimes \mathbb{C}(\beta))) \arrow[l] \arrow[r] \arrow[u] \arrow[d, equal] & \operatorname{Spec}(R[z] \otimes \mathbb{C}(\beta)) \arrow[r] \arrow[u] \arrow[d, equal] & \beta \arrow[u] \arrow[d, equal] \\
OH_{(k_1,\dots,k_r)}(\phi) & OH_{(k_1,\dots,k_r)}(\phi) \times_{\operatorname{Spec} R} \operatorname{Spec} R_b \arrow[l] \arrow[r] \arrow[d, equal] & \operatorname{Spec} R_b \arrow[r] & \beta \\
OH_{(k_1,\dots,k_r)}(\phi_b) \arrow[u] \arrow[r, equal] & OH_{(k_1,\dots,k_r)}(\phi_b). & &
\end{tikzcd}$$

□

We can now proceed with the proof, by induction on $k$, of Theorem 1.6 (which we recall).

**Theorem 4.2.** *(Aligned Ordered Hilbert Scheme Theorem) Let $X$ be a smooth connected dimension $n$ quasi-projective variety embedded in $\mathbb{P}^N$, with $N = n + c$.*

*For $k = k_1 + ... + k_r$, with $k_i > 0$, let $OH_{(k_1,\dots,k_r)}(X)$ be the ordered Hilbert scheme of aligned, finite, degree $k$ subschemes of $X$, with (ordered) multiplicities $k_i$ at the ordered points $x_i$ (possibly coinciding). Consider the natural projective line bundle $H_{(k_1,\dots,k_r)}(X) \times_G \mathcal{I}$ over $H_{(k_1,\dots,k_r)}(X)$ and the projection*

$$\theta_{(k_1,\dots,k_r)} : OH_{(k_1,\dots,k_r)}(X) \times_G \mathcal{I} \to \mathcal{I} \to \mathbb{P}^N.$$

*The general fiber of $\theta_{(k_1,\dots,k_r)}$ is smooth of dimension $N-1+r-kc$.*

We apply the results of the preceding section in the the case $Y=X$ and we claim that Theorem 1.6 is a consequence of the next proposition. We go on considering a point $\{Z\}=(L,a_1,...,a_r)\in OH_{(k_1,\dots,k_r)}(X)$ corresponding to a finite scheme $Z\subset L\cap X$ with multiplicities $(k_1,...,k_r)$ at the distinct points

$$(a_1,...,a_r)\in \operatorname{Spec}\mathbb{C}[z]\cap X\subset L\cap X.$$

**Proposition 4.3.** *Consider $\{Z\}=(L,a_1,...,a_r)\in OH_{(k_1,\dots,k_r)}(X)$, with $k=\sum k_i>1$.*

*For $k_r=1$, define $\{Z'\}=(L,a_1,...,a_{r-1})\in OH_{(k_1,\dots,k_{r-1},k_{r-1})}(X)$, where $Z'\subset Z$ is the finite, degree $k-1$, subscheme of $Z$ with multiplicity $k_i$ at $a_i$ for $i\le r-1$ and multiplicity $0=k_r-1$ at $a_r$.*

*For $k_r>1$, define $\{Z'\}=(L,a_1,...,a_r)\in OH_{(k_1,\dots,k_r-1)}(X)$, where $Z'\subset Z$ is the finite, degree $k-1$, subscheme of $Z$ with multiplicity $k_i$ at $a_i$ for $i\le r-1$ and multiplicity $k_r-1$ at $a_r$.*

*If $OH_{(k_1,\dots,k_r)}(X)$ is not smooth of dimension $2N-2+r-kc$ at $\{Z\}$ , then*
*- for $k_r=1$, the point $(\{Z'\},\beta)\in OH_{(k_1,\dots,k_{r-1})}(X)\times_G\mathcal{I}$ is a point of ramification for $\theta_{(k_1,\dots,k_{r-1})}$,*
*- for $k_r>1$, the point $(\{Z'\},\beta)\in OH_{(k_1,\dots,k_r-1)}(X)\times_G\mathcal{I}$ is a point of ramification for $\theta_{(k_1,\dots,k_r-1)}$.*

*Proof* of Theorem 1.6.

We assume that the proposition is true and we proceed by induction on $k$. Note that for $k=1$ the aligned ordered Hilbert scheme $OH_{(1)}(f)$ is smooth, hence $OH_{(1)}(X)\times_G\mathcal{I}$ is smooth and the general fiber of $\phi_{(1)}$ is smooth of dimension $N-1+1-c=n$ by Bertini's Theorem.

Let $k$ be minimum number for which there exists a partition $k=k_1+...+k_r,\quad k_i>0$ and such that the generic fiber of $\phi_{(k_1,\dots,k_r)}$ fails to be smooth of dimension $N-1+r-kc$. By Bertini's theorem, this implies that the inverse image of the singular locus of $OH_{(k_1,\dots,k_r)}(X)$ in $OH_{(k_1,\dots,k_r)}(X)\times_G\mathcal{I}$ dominates $\mathbb{P}^N$. Applying proposition 4.3, we find that the ramification locus of $\theta_{(k_1,\dots,k_{r-1})}$ (or $\theta_{(k_1,\dots,k_r-1)}$ if $k_r>1$) dominates $\mathbb{P}^N$. This contradicts the minimality of $k$.

□

*Proof* of Proposition 4.3.

Note that by Proposition 2.3 we can assume $a_i\neq a_j$ for $i\neq j$ (this is a key point!).

If $OH_{(k_1,\dots,k_r)}(\phi)$ is not smooth of dimension $2N-2+r-kc$ at $x=(L,a_1,...,a_r)$, then, by Proposition 3.2, the elements

$$\sum_i u_iq_{1,i}^{(s-1)}(a_j)+\sum_i(u_ia_j+v_i)q_{1,i}^{(s)}(a_j),\quad j=1,...,r,\quad 0\le s\le h_j-1,$$

and

$$\sum_i u_iq_{t,i}^{(s-1)}(a_j)+\sum_i(u_ia_j+v_i)q_{t,i}^{(s)}(a_j),\quad t>1,\quad j=1,...,r,\quad 0\le s\le k_j-1$$

are linearly dependent in $\mathcal{M}/\mathcal{M}^2$.

Specializing in $\mathcal{M}_b=\mathcal{M}/(u_ib+v_i)$, we see that the elements

$$\sum_i \bar{u}_i[q_{1,i}^{(s-1)}(a_j)+(a_j-b)q_{1,i}^{(s)}(a_j)],\quad j=1,...,r,\quad 0\le s\le h_j-1,$$

$$\sum_i \bar{u}_i[q_{t,i}^{(s-1)}(a_j)+(a_j-b)q_{t,i}^{(s)}(a_j)],\quad t>1,\quad j=1,...,r,\quad 0\le s\le k_j-1$$

are linearly dependent in $\mathcal{M}_b/\mathcal{M}_b^2$.

In the special case $b=a_r$, we find that the elements

$$1)\quad \sum_i \bar{u}_i[q_{1,i}^{(s-1)}(a_j)+(a_j-a_r)q_{1,i}^{(s)}(a_j)],\quad j=1,...,r-1,\quad 0\le s\le h_j-1,$$

or equivalently

$$\sum_i \bar{u}_i[q_{1,i}^{(s)}(a_j)], \quad j=1,...,r-1, \quad 0 \leq s \leq h_j-1,$$

$$2) \quad \sum_i \bar{u}_i[q_{1,i}^{(s-1)}(a_r)], \quad 0 \leq s \leq h_r-1,$$

or equivalently

$$\sum_i \bar{u}_i[q_{1,i}^{(s)}(a_r)], \quad 0 \leq s \leq h_r-2=(h_r-1)-1,$$

$$3) \sum_i \bar{u}_i[q_{t,i}^{(s-1)}(a_j)+(a_j-a_r)q_{t,i}^{(s)}(a_j)], \quad t>1, \quad j=1,...,r-1, \quad 0 \leq s \leq k_j-1,$$

or equivalently

$$\sum_i \bar{u}_i[q_{t,i}^{(s)}(a_j)], \quad t>1, \quad j=1,...,r-1, \quad 0 \leq s \leq k_j-1,$$

and

$$4) \quad \sum_i \bar{u}_i q_{t,i}^{(s-1)}(a_r), \quad t>1, \quad 0 \leq s \leq k_j-1,$$

or equivalentely

$$\sum_i \bar{u}_i q_{t,i}^{(s)}(a_r), \quad t>1, \quad 0 \leq s \leq k_r-2=(k_r-1)-1$$

are linearly dependent in $\mathcal{M}_{a_r}/\mathcal{M}_{a_r}^2$.

Using then Corollary 3.6 and Proposition 4.1, one sees easily that

$$OH_{(k_1,...,k_{r-1})}(\phi_b)=\theta^{-1}_{(k_1,...,k_{r-1})}(\beta)\cap(OH_{(k_1,...,k_{r-1})}(\phi)\times_{\text{Spec }R}(\text{Spec }R[z]))$$

is singular at $\{Z'\}$ when $k_r=1$, and that

$$OH_{(k_1,...,k_r-1)}(\phi_b)=\theta^{-1}_{(k_1,...,k_r-1)}(\beta)\cap(OH_{(k_1,...,k_r-1)}(\phi)\times_{\text{Spec }R}(\text{Spec }R[z]))$$

is singular at $\{Z'\}$ when $k_r>1$.

□

As we already remarked, Proposition 4.3 implies Theorem 1.6 which in turn implies Theorem 1.3 which yields Theorem 1.1.

## 5. Examples, questions and conjectures

Examples

**Example 5.1.** *As a first example, consider a projected* Veronese surface $X\subset\mathbb{P}^4$ *(yes projected in* $\mathbb{P}^4$*), and a general projection* $X\to X_1\subset\mathbb{P}^3$.

The Steiner surface $X_1$ is well known. We describe its singular locus.
- $X_2=X_{\{1,1\}}$ is composed of three lines through a point $x$ and not in a plane.

The normalization $\tilde{X}_2$ of $X_2$, a fiber of the map $\phi_{\{1,1\}}: H_{\{1,1\}}(X)\times_G\mathcal{I}\to\mathcal{I}\to\mathbb{P}^4$, is a disjoint union of three lines.
- $X_2$ has a closed subscheme $X_{\{2\}}$ composed of 6 distinct pinch points, 2 on each of the 3 lines.
- The degree 1 finite scheme $X_3=X_{\{1,1,1\}}=\{x\}$ is the triple locus of $X_1$, as well as the singular and triple locus of $X_2$. We note that, as stated in Theorem 1.1, we have $X_{\{2,1\}}=\emptyset$, in other words $X_{\{2\}}$ and $X_3$ are disjoint.

**Example 5.2.** *The Veronese surface $X \subset \mathbb{P}^4$ is one of the four* Severi Varieties. *According to a celebrated result of F. Zak (see 5.5), if $X^n \subset \mathbb{P}^N$ is a nondegenerate, dimension $n$, smooth variety with $N \leq 3n/2+1$, then $X$ is linearly complete except for the four projected Severi varieties, for which $n = 2^k$ with $k = 1,2,3,4$ and $N = 3n/2+1$.*

We consider a projected Severi variety $X \subset \mathbb{P}^{(3n/2)+1}$ and we describe the singularities of a general projection $X \to X_1 \subset \mathbb{P}^{3n/2}$.
- $X_2 = X_{\{1,1\}}$ is composed of three $\mathbb{P}^{n/2}$ through a point $x$ and not in a hyperplane. Its normalization $\tilde{X}_{\{1,1\}}$ is a disjoint union of three $\mathbb{P}^{n/2}$. We recall that $\tilde{X}_{\{1,1\}}$ is a general fiber of the map $\phi_{\{1,1\}} : H_{\{1,1\}}(X) \times_G \mathcal{I} \to \mathcal{I} \to \mathbb{P}^N$.
- $X_2$ has a closed subscheme $X_{\{2\}}$ composed of three disjoint quadrics of dimension $(n/2)-1$, one in each $\mathbb{P}^{n/2}$.
- The degree 1 finite scheme $X_3 = X_{\{1,1,1\}} = \{x\}$ is the triple locus of $X_1$ as well as the singular and triple locus of $X_2$.
- We note once again that the degree 6 finite scheme $X_{\{2\}}$ is smooth and disjoint from $X_3$. Indeed $X_{\{2,1\}} = \emptyset$, as stated in Theorem 1.1.

**Example 5.3.** *Consider a general skew-symmetric map $6O_{\mathbb{P}^3}(-1) \to 6O^{\mathbb{P}_3}$. The cubic surface defined by its degree 3 pfaffian is smooth and equipped with a projective $\mathbb{P}^1$-bundle smoothly embedded in $\mathbb{P}^5$ as a 3-fold of degree 7, well known as the* Palatini *3-fold.*

We describe now the singularities of a general projection $X \to X_1 \subset \mathbb{P}^4$ of a Palatini 3-fold.
- $X_2 = X_{\{1,1\}}$ is an irreducible surface of degree 11 whose smooth normalization $\tilde{X}_{\{1,1\}}$ is a fiber of $\phi_{\{1,1\}} : H_{\{1,1\}}(X) \times_G \mathcal{I} \to \mathcal{I} \to \mathbb{P}^5$.
- $X_3 = X_{\{1,1,1\}}$ is the singular and the triple locus of $X_2$. It is composed of four lines through a point $x$ and generating $\mathbb{P}^4$. The normalization of $X_{\{1,1,1\}}$ is a disjoint union of four lines.
- $X_2$ contains a pinch curve $X_{\{2\}}$ of degree 22.
- The scheme $X_{\{2\}} \cap X_3 = X_{\{2,1\}}$ is the singular locus of $X_{\{2\}}$. It is composed of 24 distinct points, six on each of the four lines.
- $X_4 = \{x\}$ is a degree 1 finite scheme. By Theorem 1.1 we have $X_4 \cap X_{\{2,1\}} = X_{\{2,1,1\}} = \emptyset$.

**Example 5.4.** *Consider an elliptic quintic ruled surface $S \subset \mathbb{P}^4$ (the lines of $S$ are parametrized by a section of $G(1,4)$ by a general $\mathbb{P}^4$ in the Plücker space).*

We describe the singularities of a general projection $S \to S_1 \subset \mathbb{P}^3$.
- The double locus $S_{\{1,1\}} = S_2 \subset S_1$ is a smooth quintic elliptic curve.
- The triple locus $S_3 = S_{\{1,1,1\}}$ is empty. This deserves a comment, see Theorem 5.6.
- There are ten distinct pinch points on $S_2$, i.e. $S_{\{2\}}$ is a smooth, degree 10, finite scheme.

Questions and conjectures

Our first and main question is classical.
*Let $X \subset \mathbb{P}^N$ be a dimension $n$ smooth variety, not contained in a hypersurface of degre $< k$. For which $(n, N, k)$ do the $k$-secant lines to $X$ fill up the space?*

We know by Theorem 1.3 that $k(N-n-1) \leq N-1$ is a necessary consequence.

For $k = 2$ the complete answer was given by F. Zak (see for example [9] or [6]).

**Theorem 5.5.** *(F. Zak) 1) If $N-1-3(N-n-1) \geq -2$, then the 2-secant lines to a nondegenerate dimension $n$ smooth variety $X \subset \mathbb{P}^N$ fill up the ambient space except for the four (nonprojected) Severi Varieties, in which case $n = 2^k$ with $k = 1,2,3,4$ and $N-1-3(N-n-1) = -2$ (i.e. $N = 3n/2+2 = 3.2^{k-1}+2$).*

It is not irrelevant to recall that the 2-secant lines to a Severi variety $X$ fill up a cubic hypersurface of $\mathbb{P}^N$.

Note also that the nonprojected Severi Varieties are cut out by quadric hypersurfaces.

For $k = 3$, the question is open except for $N = 4$, in which case A. Aure proved the following result (see [2]):

**Theorem 5.6.** *(A. Aure) Elliptic quintic scrolls in $\mathbb{P}^4$ are the only smooth surfaces not contained in a quadric hypersurface whose 3-secant lines do not fill up $\mathbb{P}^4$.*

The 3-secant lines to a quintic elliptic surface fill up a quintic hypersurface of $\mathbb{P}^4$.

Note that elliptic quintic scrolls in $\mathbb{P}^4$ are cut out by cubic hypersurfaces.

This suggests

**Conjecture 5.7.** *There exists a function $\phi(k)$ such that for any dimension $n$,smooth variety $X \subset \mathbb{P}^N$ not contained in a hypersurface of degree $< k$ sone has:*

*1) if $N - 1 - (k+1)(N - n - 1) > \phi(k)$, then the $k$-secant lines to $X$ fill up the ambient space;*

*2) if $N - 1 - (k+1)(N - n - 1) = \phi(k)$ and the $k$-secant lines to $X$ do not fill up the ambient space, then $X$ is cut out by hypersurfaces of degree $k$.*

From Zak's Theorem we get $\phi(2) = -2$ and basing on Aure's Theorem we conjecture $\phi(3) = -1$.

Our second series of questions concerns the irreducibility of the loci $X_k$ of a general projection of a smooth variety.

We begin with recalling Franchetta's famous theorem.

**Theorem 5.8.** *(A. Franchetta) The Veronese surface in $\mathbb{P}^4$ is the only smooth projective surface whose general projection to $\mathbb{P}_3$ has a reducible double locus.*

Of course, Franchetta does not assume $S \subset \mathbb{P}^4$, but by Bertini's Theorem this is the only difficult case.

As a comment to this result, we recall that

- if $X$ is a projected Severi variety $X^n \subset \mathbb{P}^{(3n/2)+1}$, then the locus $X_2$ is a union of three $\mathbb{P}^{n/2}$ through a point (Example 5.2),
- if $X$ is a Palatini 3-fold, then the locus $X_3$ is a union of four lines through a point (Example 5.3),
- there exists a dimension 6 smooth variety $X \subset \mathbb{P}^9$ whose general projection has a reducible triple locus. $X_3$ is a union of four planes passing through a point $x$, with $X_4 = \{x\}$).

We dare a bold conjecture

**Conjecture 5.9.** *Let $X \subset \mathbb{P}^N$ be a dimension $n$, smooth irreducible variety.*

*1) If $(k+1)(N - n - 1) < (N - 1)$ then the locus $X_k$ of a general projection of $X$ is irreducible.*

*2) If $(k+1)(N - n - 1) = (N - 1)$ and the locus $X_k$ of a general projection of $X$ is reducible, then the finite scheme $X_{k+1}$ has degree 1 and $X_k$ is a union of $k+1$ linear space $\mathbb{P}^{N-n-1}$ passing through the point of $X_{k+1}$.*

Our conjecture is related to the following conjecture of F. Zak ([10]):

**Conjecture 5.10.** *Let $X^n \subset \mathbb{P}^N$ with $N - 1 \geq (k+1)(N - n - 1)$ be a nondegenerate irreducible variety (non necessarily smooth). Consider a general projection $X \to \mathbb{P}^{N-1}$. Then the locus $X_k \subset \mathbb{P}^{N-1}$ of points whose fiber has degree $\geq k$) is connected.*

To conclude this paper, we note here a theorem, a related conjecture r and a remark:

**Theorem 5.11.** *(F. Zak, (Theorem 1, [10]))*

*Let $X \subset \mathbb{P}^N$ be a nondegenerate irreducible variety (not necessarily smooth). Consider a general projection $X \to \mathbb{P}^{N-1}$. If the quasi-projective locus $X_k - X_{k+1}$ is connected and nonempty, then the hypersurfaces of degree $< k$ cut complete linear system on $X$.*

**Conjecture 5.12.** *Let $X^n \subset \mathbb{P}^N$ be a dimension $n$, smooth irreducible variety. The following conditions are equivalent:*

*1) the locus $X_k$ of a general projection of $X$ is reducible,*

*2) the linear system cut out by the hypersurfaces of degree $k - 1$ on $X$ is not complete.*

**Remark 5.13.** *The Palatini threefold (Example 5.3) is not quadratically normal. The second author conjectured, at a Trento conference in 1988, that this is the only non quadratically normal smooth threefold in $\mathbb{P}^5$.*

## References

[1] A. Alzati, G. Ottaviani. The Theorem of Mather on generic projections in the setting of algebraic geometry. Manuscripta Math. 74 (1992), 391-412.

[2] A. Aure. The smooth surfaces in $\mathbb{P}^4$ without apparent triple points. Duke Math. J. 57 (1988), no. 2, 423–430.

[3] R. Beheshti, D. Eisenbud. Fibers of generic projections. Compos. Math. 146 (2010), no. 2, 435–456.

[4] S. Kleiman, J. Lipman, B. Ulrich. The source double-point cycle of a finite map of codimension one. Complex projective geometry (Trieste, 1989/Bergen, 1989), 199–212, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[5] S. Kleiman, J. Lipman, B. Ulrich. The multiple-point schemes of a finite curvilinear map of codimension one. Ark. Mat. 34 (1996), no. 2, 285–326.

[6] R. Lazarsfeld, A. Van de Ven. Topics in the geometry of projective space. Recent work of F. L. Zak. With an addendum by Zak. DMV Seminar, 4. Birkhäuser Verlag, Basel, 1984.

[7] J. Mather. Generic projections. Ann. of Math. 98 (1973), 226-245.

[8] Z. Ran. The (dimension +2)-secant lemma. Invent. Math. 106 (1991), no. 1, 65–71.

[9] F.L. Zak. Tangents and secants of algebraic varieties. Translated from the Russian manuscript by the author. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993

[10] F.L. Zak. Multisecants and normality. Preprint.

Laboratoire de Mathématques de Versailles (UMR 8100 CNRS), 45, avenue des États-Unis, F-78035 Versailles cedex France

*E-mail address*: laurent.gruson@math.uvsq.fr

Institut de Mathématiques de Jussieu (UMR 7586 CNRS), Université Pierre et Marie Curie, Case 247, 4, place Jussieu, 75252 Paris Cedex 05, FRANCE

*E-mail address*: peskine@math.jussieu.fr